Average Value of a Function

We know how to find the average value given a set of numbers. Apply this knowledge to find the average value of a function over a given interval. First, we want to average the y-values, so for a finite number of y-values,

$$y_{ave} = \frac{y_1 + y_2 + y_3 + \ldots + y_n}{n}$$

$$= \frac{f\left(x_1^*\right) + f\left(x_2^*\right) + f\left(x_3^*\right) + \ldots + f\left(x_n^*\right)}{n}$$

Since $\Delta x = \dfrac{b-a}{n}$, we can solve for n, obtaining $n = \dfrac{b-a}{\Delta x}$

By substitution,

$$y_{ave} = \frac{f\left(x_1^*\right)+f\left(x_2^*\right)+f\left(x_3^*\right)+\ldots+f\left(x_n^*\right)}{n}$$

$$= \frac{f\left(x_1^*\right)+f\left(x_2^*\right)+f\left(x_3^*\right)+\ldots+f\left(x_n^*\right)}{\left(\frac{b-a}{\Delta x}\right)}$$

$$= \frac{f\left(x_1^*\right)+f\left(x_2^*\right)+f\left(x_3^*\right)+\ldots+f\left(x_n^*\right)}{b-a}\Delta x$$

$$= \frac{1}{b-a}\left[f\left(x_1^*\right)+f\left(x_2^*\right)+f\left(x_3^*\right)+\ldots+f\left(x_n^*\right)\right]\Delta x$$

Since we want to know the average value of y, we want $n \to \infty$

The result is $$y_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

The Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

To see the geometric meaning of this theorem, theorems.gsp

Find the average value of the function $f(x) = x^3 - 2x^2 + 3x - 2$ on [1,5].

$$f_{ave} = \frac{1}{5-1}\int_1^5 (x^3 - 2x^2 + 3x - 2)\,dx$$

$$= \frac{1}{4}\left(\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{3}{2}x^2 - 2x\right)\Big]_1^5$$

$$= \frac{76}{3}$$

Find the average value of $f(x)=\sqrt{x}$ on [0,4] and then find the value(s) of c such that

$$f_{ave}=f(c)$$

$$f_{ave}=\frac{1}{4-0}\int_0^4 x^{\frac{1}{2}}\,dx$$

$$=\frac{1}{4}\cdot\frac{2}{3}x^{\frac{3}{2}}\Big]_0^4$$

$$=\frac{1}{6}\cdot 4^{\frac{3}{2}}-\frac{1}{6}\cdot 0^{\frac{3}{2}}$$

$$=\frac{4}{3}$$

$$\frac{4}{3}=\sqrt{c}$$

$$\frac{16}{9}=c$$