The Chain Rule

Recall the homework problem #55 on page 189 where we

found the derivative of $y = \left[f(x)\right]^3$ to be $\frac{dy}{dx} = 3\left[f(x)\right]^2 f'(x)$

This leads us to an intuitive understanding of the chain rule, which states that if we have two differentiable functions *f* and *g*, the composition of those functions, $y = f \circ g(x) = f\big(g(x)\big)$, has the derivative

$$\frac{dy}{dx} = f'\big(g(x)\big)g'(x)$$

Using Leibniz notation where $y = f(u)$ and $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Find the derivative

1. $y = \cos\left(x^2\right)$ $\quad \frac{dy}{dx} = -\sin\left(x^2\right)\cdot 2x$

$$= -2x\sin\left(x^2\right)$$

2. $y = \sqrt{\tan x} = (\tan x)^{\frac{1}{2}}$ $\quad \frac{dy}{dx} = \frac{1}{2}(\tan x)^{-\frac{1}{2}}\cdot\sec^2 x$

$$= \frac{\sec^2 x}{2\sqrt{\tan x}}$$

3. $y = \left(x^3 - 4x^2 + 7x - 8\right)^{15}$

$$\frac{dy}{dx} = 15\left(x^3-4x^2+7x-8\right)^{14}\left(3x^2-8x+7\right)$$

4. $y = \left(x^3 + 4x\right)^3\left(5x^2 - 3x + 1\right)^4$

$$\frac{dy}{dx} = \left(x^3+4x\right)^3\cdot 4\left(5x^2-3x+1\right)^3(10x-3) + \left(5x^2-3x+1\right)^4\cdot 3\left(x^3+4x\right)^2\left(3x^2+4\right)$$

$$= \left(x^3+4x\right)^2\left(5x^2-3x+1\right)^3\left[4\left(x^3+4x\right)(10x-3) + 3\left(3x^2+4\right)\left(5x^2-3x+1\right)\right]$$

$$= \left(x^3+4x\right)^2\left(5x^2-3x+1\right)^3\left(85x^4-39x^3+229x^2-84x+12\right)$$

5. $y = \dfrac{\left(2+3x^2\right)^5}{(3x-7)^4}$

$$\frac{dy}{dx} = \frac{(3x-7)^4 \cdot 5(2+3x^2)^4 \cdot 6x - (2+3x^2)^5 \cdot 4(3x-7)^3 \cdot 3}{\left[(3x-7)^4\right]^2}$$

$$= \frac{6(3x-7)^3(2+3x^2)^4\left[5x(3x-7) - 2(2+3x^2)\right]}{(3x-7)^8}$$

$$= \frac{6(2+3x^2)^4(9x^2-35x-4)}{(3x-7)^5}$$

$b^2-4ac = (-35)^2 - 4(9)(-4)$

$= 1369 = 37^2$

a perfect sq. $\Rightarrow$ the trinomial factors

$$= \frac{6(2+3x^2)^4(9x+1)(x-4)}{(3x-7)^5}$$

Find the derivative of $f(x)=a^x$

a is a constant
$a>0,\ a\neq 1$

Note: $a^x=\left(e^{\ln a}\right)^x=e^{(\ln a)x}$

$f(x)=a^x=e^{(\ln a)x}$

$f'(x)=e^{(\ln a)x}\cdot \ln a$

$=a^x\cdot \ln a$

6. $y=\sqrt{2^x \sin 2x} = (2^x \sin 2x)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(2^x \sin 2x)^{-\frac{1}{2}} (2^x \cdot \cos 2x \cdot 2 + \sin 2x \cdot 2^x \cdot \ln 2)$$

$$= \frac{2^{x+1} \cos 2x + \ln 2 \cdot 2^x \sin 2x}{2\sqrt{2^x \sin 2x}}$$

or

$$= \frac{2^x (2\cos 2x + \ln 2 \cdot \sin 2x)}{2\sqrt{2^x \sin 2x}} \quad \star$$