The Product and Quotient Rules

Let $y = x^2 \cdot x^3$ Find $\frac{dy}{dx}$ using two methods: 1) by guessing a rule for the product of two functions and 2) simplifying the expression and then differentiating to yield the correct result.

1) $\frac{dy}{dx} = 2x \cdot 3x^2 = 6x^3 \longleftarrow$ incorrect

2) $y = x^2 \cdot x^3 = x^5 \longrightarrow \frac{dy}{dx} = 5x^4$

Let $y = f(x) \cdot g(x)$ Find $\frac{dy}{dx}$

The derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f(x)\frac{d}{dx}\left(g(x)\right) + g(x)\frac{d}{dx}\left(f(x)\right)$$

$$\frac{dy}{dx} = x^2 \cdot \frac{d}{dx}(x^3) + x^3 \cdot \frac{d}{dx}(x^2) = x^2 \cdot 3x^2 + x^3 \cdot 2x$$
$$= 3x^4 + 2x^4 = 5x^4 \checkmark$$

The following rule I offer without proof (the proof is shown in the textbook):

The derivative of a quotient of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\frac{d}{dx}(f(x))-f(x)\frac{d}{dx}(g(x))}{[g(x)]^2}$$

Low d High minus High d Low,
all over the square of Low.

Write each derivative rule using prime notation.

$$(cf)' = cf'$$

$$(f+g)' = f' + g'$$

$$(f-g)' = f' - g'$$

$$(fg)' = fg' + gf'$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

☆ also, for example:

$$(fg)'(x) = f(x)g'(x) + g(x)f'(x)$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$y=f(x)g(x)$$

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)\overbrace{-f(x+h)g(x)+f(x+h)g(x)}^{+0}-f(x)g(x)}{h}$$

$$=\lim_{h\to 0}\left[f(x+h)\cdot\frac{g(x+h)-g(x)}{h}+g(x)\cdot\frac{f(x+h)-f(x)}{h}\right]$$

$$=\lim_{h\to 0}f(x+h)\cdot\frac{g(x+h)-g(x)}{h}+\lim_{h\to 0}g(x)\cdot\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+\lim_{h\to 0}g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=f(x)\cdot g'(x)+g(x)\cdot f'(x)$$