Calculating Limits Using the Limit Laws

Limits for more complicated functions can be found using the limit laws.

Assuming that $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist, and that c is a constant,

1. $\lim\limits_{x\to a}\left[f(x)+g(x)\right]=\lim\limits_{x\to a} f(x)+\lim\limits_{x\to a} g(x)$

2. $\lim\limits_{x\to a}\left[f(x)-g(x)\right]=\lim\limits_{x\to a} f(x)-\lim\limits_{x\to a} g(x)$

3. $\lim\limits_{x\to a}\left[c\cdot f(x)\right]=c\cdot\lim\limits_{x\to a} f(x)$

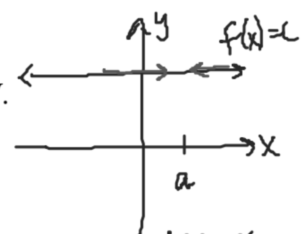
4. $\lim\limits_{x\to a}\left[f(x)g(x)\right]=\lim\limits_{x\to a} f(x)\cdot\lim\limits_{x\to a} g(x)$

5. $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ if $\lim\limits_{x\to a} g(x)\neq 0$

These Limit Laws extend to many more combinations of functions. Using the Limit Law #4 repeatedly where *f* and *g* are the same function, we get

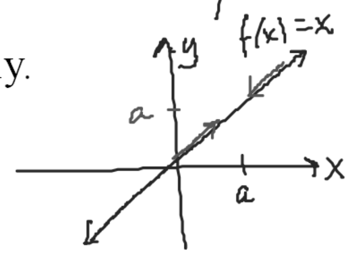
6. $\lim\limits_{x \to a} [f(x)]^n = \left[\lim\limits_{x \to a} f(x)\right]^n$ where n is a positive integer

Consider the constant function $f(x) = c$ graphically.



7. $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} c = c$

Now consider the function $f(x) = x$ graphically.



8. $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} x = a$

Using Limit Laws 6 and 8 together we get

9. $\lim\limits_{x \to a} x^n = a^n$

Similarly, 10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer and we assume that $a > 0$ when n is even.

More generally, 11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

where n is a positive integer and we assume that $\lim_{x \to a} f(x) > 0$ when n is even.

Evaluate the limit, justifying each step.

$$\lim_{x \to 3}\left(3x^2 - 4x + 2\right) = \lim_{x \to 3} 3x^2 - \lim_{x \to 3} 4x + \lim_{x \to 3} 2 \qquad \#1, \#2$$

$$= 3 \cdot \lim_{x \to 3} x^2 - 4 \cdot \lim_{x \to 3} x + 2 \qquad \#3, \#7$$

$$= 3 \cdot 3^2 - 4 \cdot 3 + 2 \qquad \#9, \#8$$

$$= 17$$

Notice that we could have gotten the limit by substitution. For polynomial and rational functions,

$$\lim_{x \to a} f(x) = f(a)$$

Rational function

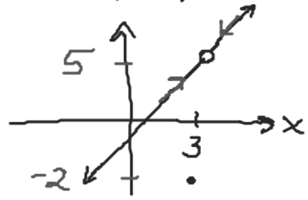
$$f(x) = \frac{P(x)}{Q(x)} \quad \text{where } P, Q \text{ are polynomials}$$

Radical functions (as well as some soon-to-be-named other functions) fit in this category, too.

(that is, that you can evaluate by substitution)

Find $\lim_{x \to 3} f(x)$ if $f(x) = \begin{cases} 2x-1, & x \neq 3 \\ -2, & x = 3 \end{cases}$

$f(3) = -2$



$= \lim_{x \to 3} (2x-1) = 2 \cdot 3 - 1$

$= 5$

Find $\lim_{h \to 0} \frac{(5+h)^2 - 25}{h} = \lim_{h \to 0} \frac{25 + 10h + h^2 - 25}{h}$

$= \lim_{h \to 0} \frac{10h + h^2}{h}$

$= \lim_{h \to 0} \frac{h(10+h)}{h}$

$= \lim_{h \to 0} (10+h)$

$= 10 + 0$

$= 10$

$(a+b)(a-b) = a^2 - b^2$

Find $\lim\limits_{r\to 0} \frac{\sqrt{r^2+4}-2}{r^2} \cdot \frac{\sqrt{r^2+4}+2}{\sqrt{r^2+4}+2}$

$= \lim\limits_{r\to 0} \frac{(r^2+4)-4}{r^2(\sqrt{r^2+4}+2)}$

$= \lim\limits_{r\to 0} \frac{r^2}{r^2(\sqrt{r^2+4}+2)}$

$= \lim\limits_{r\to 0} \frac{1}{\sqrt{r^2+4}+2}$

$= \frac{1}{\sqrt{0^2+4}+2} = \frac{1}{4}$

Find $\lim\limits_{x\to -2} g(x)$ if $g(x) = \begin{cases} \sqrt{x+2}, & x > -2 \\ 5x+10, & x < -2 \end{cases}$

$\lim\limits_{x\to -2^-} (5x+10) = 5(-2)+10 = 0$

$\lim\limits_{x\to -2^+} \left(\sqrt{x+2}\right) = \sqrt{-2+2} = 0$

Because $\lim\limits_{x\to -2^-} g(x) = \lim\limits_{x\to -2^+} g(x) = 0$,

$\therefore \lim\limits_{x\to -2} g(x) = 0$