Inverse Functions

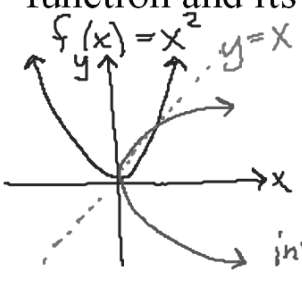
In order to find the inverse of a function, we interchange the x and y, then solve the resulting equation for y. This switches everything about the two variables: the domain of the function becomes the range of the inverse, and the range of the function becomes the domain of the inverse, for every point on the graph of the function (a, b) an image point (b, a) exists for the inverse, horizontal features for the function become vertical for the inverse, vertical features of the function become horizontal for the inverse, etc. The graphical meaning of the inverse is that the graph of the inverse is a reflection of the graph of the function over the line $y = x$. Consider graphically the squaring function and its inverse.

$$y = x^2$$

$$\downarrow \text{inv}$$

$$x = y^2$$

$$y = \pm\sqrt{x}$$



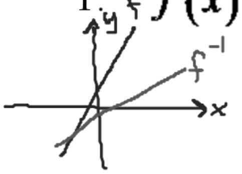
inverse (not a function – fails the vertical-line test)

In order for the inverse to be a function, the original function must be **one-to-one**, that is, for each value of *x* there can be only one value of *y* (the definition of a function), but also that for each value of *y* there is only one *x* value. One can test the first condition graphically using the *vertical line test*, and the second condition can likewise be tested using the *horizontal line test*.

We can make a function be one-to-one by restricting its domain.

Find the inverse of the functions below, making any domain restrictions needed to make the inverse be a function.

1. $\boldsymbol{f(x)=5x+2}$



$y=5x+2$

$\downarrow$ inv

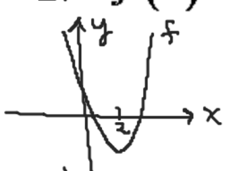
$x=5y+2$

$x-2=5y$

$y=\frac{x-2}{5}$

$\therefore f^{-1}(x)=\frac{1}{5}x-\frac{2}{5}$

2. $\boldsymbol{f(x)=x^2-4x+2}$



$h=\frac{-b}{2a}=2$

not one-to-one $\rightarrow$ restrict domain

$y=x^2-4x+2,\ x\geq 2$

$\downarrow$ inv

$x=y^2-4y+2,\ y\geq 2$

$x-2=y^2-4y$

$x-2+4=y^2-4y+4$

$x+2=(y-2)^2$

$\pm\sqrt{x+2}=y-2$

$y=2\pm\sqrt{x+2}$ but $y\geq 2$!

$\therefore f^{-1}(x)=2+\sqrt{x+2}$

If f is a one-to-one function that is continuous on an interval, then its inverse f^{-1} is also continuous.

We have now reviewed exponential functions and inverse functions. Now we'll put those two concepts together. Consider the function $f(x) = a^x$ What is its inverse?

$y = a^x$

$\downarrow$ inv

$x = a^y$

$y = \log_a x$

$\therefore f^{-1}(x) = \log_a x$

An algebraic test for inverses is that if f and g are inverses, then

$$f(g(x)) = g(f(x)) = x$$

Consequently, $\log_a(a^x) = x$ and $a^{\log_a x} = x$

Properties of Logarithms

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a\left(x^r\right) = r\log_a x$$

The natural logarithm is the inverse of the exponential function with base e, i.e.,

$$\log_e x \equiv \ln x$$

Change-of-Base Formula

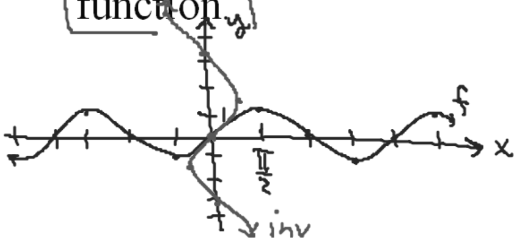
For any positive a not equal to one, $\log_a x = \dfrac{\ln x}{\ln a}$

Inverse Trigonometric Functions

Now that we have studied inverse functions, we can add the trigonometric functions to the list of functions whose inverse we can find.

Consider the sine function. Is it one-to-one? No

Graph the function and its inverse as a relation and as a function.



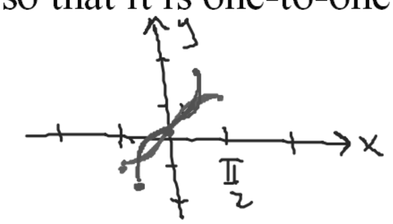
Rules for restricting the domain.

1. Retain the entire range of the function.
2. Stay as close to zero as possible.
3. Given equally good options, positive or negative, choose the positive.

What restriction can we make so that it is one-to-one?

$D_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow D_{f^{-1}} = [-1, 1]$

$R_f = [-1, 1] \longrightarrow R_{f^{-1}} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



$y = \sin x$

$\downarrow$ inv

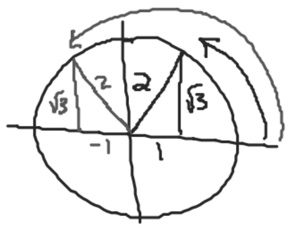
$x = \sin y$

$y = \sin^{-1} x$ or $\arcsin x$

$y = \operatorname{Sin}^{-1} x$ or $\operatorname{Arcsin} x$

Evaluate $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \theta$

$$\sin\theta = \frac{\sqrt{3}}{2}$$



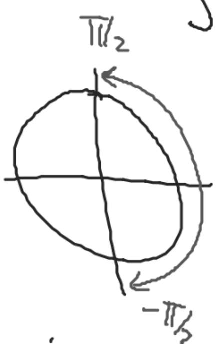
$$\theta = \frac{\pi}{3} + 2\pi n$$

or

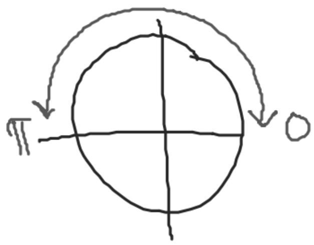
$$\frac{2\pi}{3} + 2\pi n$$

where $n \in \{\text{integers}\}$

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$



sin, csc, tan



cos, sec, cot

Evaluate $\sec\left(\text{Arcsin}\left(\frac{1}{4}\right)\right) = \sec\theta$

where $\theta = \text{Arcsin}\,\frac{1}{4}$

$$\sin\theta = \frac{1}{4} = \frac{y}{r}$$

$$\sec\theta = \frac{r}{x}$$

$$x^2 + y^2 = r^2$$

$$x^2 + 1^2 = 4^2$$

$$x^2 = 15$$

$$x = \pm\sqrt{15}$$

use $x = \sqrt{15}$

$$\therefore \sec\theta = \frac{4}{\sqrt{15}}$$

$$= \frac{4\sqrt{15}}{15}$$

$$\sec\left(\arcsin\frac{1}{4}\right) = \frac{4}{\pm\sqrt{15}}$$

$$= \pm\frac{4\sqrt{15}}{15}$$